INVOLUTIONS ON A SURFACE OF GENERAL TYPE WITH $p_g = q = 0,\ K^2 = 7$

YONGNAM LEE AND YONGJOO SHIN

Abstract. In this paper we study on the involution on minimal surfaces of general type with $p_g = q = 0$ and $K^2 = 7$. We focus on the classification of the birational models of the quotient surfaces and their branch divisors induced by an involution.

1. Introduction

In the 1930s Campedelli [3] constructed the first example of a minimal surface of general type with $p_g = 0$ using a double cover. He used a double cover of $\mathbb{P}^2$ branched along a degree 10 curve with six points, not lying on a conic, all of which are a triple point with another infinitely near triple point. After his construction, the covering method has been one of main tools for constructing new surfaces.

Surfaces of general type with $p_g = q = 0, K^2 = 1$, and with an involution have studied by Keum and the first author [8], and completed later by Calabri, Ciliberto and Mendes Lopes [1]. Also surfaces of general type with $p_g = q = 0, K^2 = 2$, and with an involution have studied by Calabri, Mendes Lopes, and Pardini [2]. Previous studies motivate the study of surfaces of general type with $p_g = q = 0, K^2 = 7$, and with an involution.

We know that a minimal surface of general type with $p_g = q = 0$ satisfies $1 \leq K^2 \leq 9$. One can ask whether there is a minimal surface of general type with $p_g = q = 0$, and with an involution whose quotient is birational to an Enriques surface. Indeed, there are examples that are minimal surfaces of general type with $p_g = q = 0$, and $K^2 = 1, 2, 3, 4$ constructed by a double cover of an Enriques surface in [6], [8], [9], [14]. On the other hand, there are no minimal surfaces of general type with $p_g = q = 0$ and $K^2 = 9$ (resp. 8) having an involution whose quotient is birational to an Enriques surface by Theorem 4.3 (resp. 4.4) in [5]. In the cases $K^2 = 3$ and 4, eight nodes on an Enriques surface are used to construct a double cover. Since an Enriques surface has at most eight nodes and one already use these eight nodes when one construct a surface with $K^2 = 3, 4$, it is reasonable guess that the quotient is not birational to an Enriques surface in the cases $K^2 = 5, 6, 7$. We cannot give the affirmative answer for the case $K^2 = 7$. But we have only two possible cases by excluding all other cases. Precisely, we prove the following in Section 4.

2000 *Mathematics Subject Classification.* 14J29.

Theorem. *Let S be a minimal surface of general type with $p_g(S) = q(S) = 0$, $K_S^2 = 7$ having an involution σ. Suppose that the quotient S/σ is birational to an Enriques surface. Then the number of fixed points is 9, and the fixed divisor is a curve of genus 3 or consists of two curves of genus 1 and 3. Furthermore, S has a 2-torsion element.*

Let S be a minimal surface of general type with $p_g(S) = q(S) = 0$ having an involution σ. There is a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\epsilon"] \arrow[d, "\tilde{\pi}"'] & S \arrow[d, "\pi"] \\ W \arrow[r, "\eta"] & \Sigma \end{tikzcd}$$

In this diagram π is the quotient map induced by the involution σ. And ϵ is the blow-up of S at k isolated fixed points of σ. Also, $\tilde{\pi}$ is induced by the quotient map π and η is the minimal resolution of the k double points made by the quotient map π. And, there is a fixed divisor R of σ on S which is the union of a smooth, possibly reducible, curve. We set $R_0 := \epsilon^*(R)$ and $B_0 := \tilde{\pi}(R_0)$. Let Γ_i be an irreducible component of B_0. When we write $\underset{(m,n)}{\Gamma_i}$, m means $p_a(\Gamma_i)$ and n is Γ_i^2.

In the paper, we give the classification of the birational models of the quotient surfaces and their branch divisors induced by an involution when $K_S^2 = 7$. Precisely, we have the following table of classification.

k	K_W^2	B_0	W
5	2	$\underset{(1,-2)}{\Gamma_0}$	minimal of general type
7	1	$\underset{(3,2)}{\Gamma_0}$	minimal of general type
7	0	$\underset{(2,-2)}{\Gamma_0}$ $\underset{(2,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$	minimal properly elliptic, or of general type whose the minimal model has $K^2 = 1$
9	-2	$\underset{(4,2)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1}$ $\underset{(3,-2)}{\Gamma_0}$ $\underset{(4,4)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$ $\underset{(4,4)}{\Gamma_0} + \underset{(0,-6)}{\Gamma_1}$ $\underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ $\underset{(3,2)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$ $\underset{(3,2)}{\Gamma_0} + \underset{(1,-4)}{\Gamma_1}$ $\underset{(2,-2)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1}$ $\underset{(3,2)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$ $\underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$	$\kappa(W) \leq 1$, and if W is birational to an Enriques surface then $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ or $\underset{(3,-2)}{\Gamma_0}$.
11	-4		rational surface

If $k = 11$, the bicanonical map is composed with the involution. We will omit the classification of B_0 for $k = 11$ because there are detailed studies in [1] and [11].

The paper is organized as follows: in Section 3 we make tables of classifications of branch divisors B_0, and birational models of quotient surfaces W for each possible k; in Section 4 we study when W is birational to an Enriques surface; in Section 5 we provide some examples. The existence of W with $\kappa(W) \geq 0$ is an open question.

2. Notation and Conventions

In this section we fix the notation which will be used in this paper. In this paper, we work over the field of complex numbers.

Let X be a smooth projective surface and Γ be a curve in X. Let $\hat{\Gamma}$ is the normalization of Γ. We set:

K_X: a canonical divisor of X;
$NS(X)$: the Néron-Severi group of X;
$\rho(X)$: the rank of $NS(X)$;
$\kappa(X)$: the Kodaira dimension of X;
$q(X)$: the irregularity of X, that is, $h^1(X, \mathcal{O}_X)$;
$p_g(X)$: the geometric genus of X, that is, $h^0(X, \mathcal{O}_X(K_X))$;
$p_a(\Gamma)$: the arithmetic genus of Γ, that is, $\Gamma(\Gamma + K_X)/2 + 1$;
$p_g(\Gamma)$: the geometric genus of Γ, that is, $h^0(\hat{\Gamma}, \mathcal{O}_{\hat{\Gamma}}(K_{\hat{\Gamma}}))$;
$\equiv$: the linear equivalence of divisors on a surface;
$\sim$: the numerical equivalence of divisors on a surface;
Γ: (m, n) or $\underset{(m,n)}{\Gamma}$: m is $p_a(\Gamma)$ and n is the self intersection number of Γ;
We usually omit the sign $\cdot$ of the intersection product of two divisors on a surface.

Let S be a minimal surface of general type with $p_g(S) = q(S) = 0$ having an involution σ. Then there is a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\epsilon"] \arrow[d, "\tilde{\pi}"'] & S \arrow[d, "\pi"] \\ W \arrow[r, "\eta"] & \Sigma \end{tikzcd}$$

In the above diagram π is the quotient map induced by the involution σ. And ϵ is the blow-up of S at k isolated fixed points arising from the involution σ. Also, $\tilde{\pi}$ is induced by the quotient map π and η is the minimal resolution of the k double points made by the quotient map π. We denote the k disjoint (-1)-curves on V (resp. the k disjoint (-2)-curves on W) related to the k disjoint isolated fixed points on S (resp. the k double points on Σ) as E_i (resp. N_i), $i = 1, \ldots, k$. And, there is a fixed divisor R of σ on S which is the union of a smooth, possibly reducible, curve. So we set $R_0 := \epsilon^*(R)$ and $B_0 := \tilde{\pi}(R_0)$.

The map $\tilde{\pi}$ is a flat double cover branched on $\tilde{B} := B_0 + \sum_{i=1}^{k} N_i$. Thus there exists a divisor L on W such that $2L \equiv \tilde{B}$ and

$$\tilde{\pi}_* \mathcal{O}_V = \mathcal{O}_W \oplus \mathcal{O}_W(-L).$$

Moreover, $K_V \equiv \tilde{\pi}^*(K_W + L)$ and $K_S \equiv \pi^* K_\Sigma + R$.

3. Classification of branch divisors and quotient surfaces

In this section we classify the possibilities of branch divisors B_0 and the birational models of W with respect to the number of isolated fixed points and K_W^2.

Since $\epsilon^*(2K_S) \equiv \tilde{\pi}^*(2K_W + B_0)$, the divisor $2K_W + B_0$ is nef and big, and $(2K_W + B_0)^2 = 2K_S^2$. First we recall the results in [1] and [5].

Proposition 3.1 (Proposition 3.3 and Corollary 3.5 in [1])**.** *Let S be a minimal surface of general type with $p_g = 0$ and let σ be an involution of S. Then*
(i) $k \geq 4$*;*
(ii) $K_W L + L^2 = -2$*;*
(iii) $h^0(W, \mathcal{O}_W(2K_W + L)) = K_W^2 + K_W L$*;*
(iv) $K_W^2 + K_W L \geq 0$*;*
(v) $k = K_S^2 + 4 - 2h^0(W, \mathcal{O}_W(2K_W + L))$*;*
(vi) $h^0(W, \mathcal{O}_W(2K_W + B_0)) = K_S^2 + 1 - h^0(W, \mathcal{O}_W(2K_W + L))$*;*
(vii) $K_W^2 \geq K_V^2$.

Proposition 3.2 (Corollary 3.6 in [1])**.** *Let S be a minimal surface of general type with $p_g = 0$, let $\varphi \colon S \to \mathbb{P}^{K_S^2}$ be the bicanonical map of S and let σ be an involution of S. Then the following conditions are equivalent:*
(i) φ is composed with σ;
(ii) $h^0(W, \mathcal{O}(2K_W + L)) = 0$*;*
(iii) $K_W(K_W + L) = 0$*;*
(iv) the number of isolated fixed points of σ is $k = K_S^2 + 4$.

By (i) and (v) of Proposition 3.1, the possibilities of k are $5, 7, 9, 11$ if $K_S^2 = 7$. In particular, if $k = 11$, the bicanonical map φ is composed with the involution, which is treated by Proposition 3.2.

Lemma 3.3 (Theorem 3.3 in [5])**.** *Let W be a smooth rational surface and let $N_1, \ldots, N_k \subset W$ be disjoint nodal curves. Then*
(i) $k \leq \rho(W) - 1$, and equality holds if and only if $W = \mathbb{F}_2$;
(ii) if $k = \rho(W) - 2$ and $\rho(W) \geq 5$, then k is even.

Lemma 3.4 (Proposition 4.1 in [5] and Remark 4.3 in [7])**.** *Let W be a surface with $p_g(W) = q(W) = 0$ and $\kappa(W) \geq 0$, and let $N_1, \ldots, N_k \subset W$ be disjoint nodal curves. Then*
(i) $k \leq \rho(W) - 2$ unless W is a fake projective plane;
(ii) $k = \rho(W) - 2$, then W is minimal unless W is the blow-up of a fake projective plane at one point or at two infinitely near points.

Denote $D := 2K_W + B_0$ for convenience. For each k, we get the following.

Theorem 3.5. *Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 7$ having an involution σ. Then*
(i) $D^2 = 14$;
(ii) If $k = 11$, then $K_W D = 0$, $K_W^2 = -4$, and W is a rational surface;
(iii) If $k = 9$, then $K_W D = 2$, $K_W^2 = -2$, and $\kappa(W) \leq 1$;
(iv) If $k = 7$, then $K_W D = 4$, $0 \leq K_W^2 \leq 1$, and $\kappa(W) \geq 1$. Furthermore, if W is properly elliptic then $K_W^2 = 0$, and if $K_W^2 = 1$ then W is minimal of general type. And if $K_W^2 = 0$ and W is of general type then $K_{W'}^2 = 1$ where W' is the minimal model of W;
(v) If $k = 5$, then $K_W D = 6$, $K_W^2 = 2$, and W is minimal of general type.

Proof. (i) It is obtained by $\epsilon^*(2K_S) \equiv \tilde{\pi}^*(D)$ and $K_S^2 = 7$.
(ii) Firstly, $K_W D = 0$ because $K_W D = 2K_W(K_W + L) = 0$ by Proposition 3.2. Secondly, $K_V^2 = K_S^2 - k = 7 - 11 = -4$. So $K_W^2 \geq -4$ by (vii) of Proposition 3.1. Finally, $K_W^2 \leq 0$ by the algebraic index theorem because $K_W D = 0$ and D is nef and big. Since $K_W D = 0$, W can only be a rational surface or birational to an Enriques surface. Enriques surface is excluded by Theorem 3 in [16]. Also by Lemma 3.3, $k \leq \rho(W) - 3$, and so $\rho(W) \geq 14$. Therefore $K_W^2 = -4$.
(iii) Firstly, $K_W D = 2$ because $K_W D = 2K_W(K_W + L) = 2$ by (iii) and (v) of Proposition 3.1. Secondly, $K_V^2 = K_S^2 - k = 7 - 9 = -2$. So $K_W^2 \geq -2$ by (vii) of Proposition 3.1. Finally, $0 \geq (7K_W - D)^2 = 49K_W^2 - 14K_W D + D^2 = 49K_W^2 - 14$ by the algebraic index theorem because D is nef and big. So $K_W^2 \leq 0$.

If W is a rational surface then by Lemma 3.3 $k \leq \rho(W) - 3$, and so $\rho(W) \geq 12$. Therefore $K_W^2 = -2$. If $\kappa(W) \geq 0$ then by Lemma 3.4 $\rho(W) \geq 11$. If $\rho(W) = 11$ then W is minimal, so it gives a contradiction because $K_W^2 = -1$. Therefore $\rho(W) = 12$ and $K_W^2 = -2$.

Moreover, W is not of general type: Suppose W is of general type. Then we consider a birational morphism $t\colon\ W \to W'$ such that W' is the minimal model of W. Also, we can write $K_W \equiv t^*(K_{W'}) + E$, $E > 0$ since $K_W^2 \leq 0$.

Then $Dt^*(K_{W'}) = 2$; Firstly, $Dt^*(K_{W'}) \leq 2$ because $2 = DK_W = Dt^*(K_{W'}) + DE$ and D is nef. Secondly, $Dt^*(K_{W'}) \geq 2$ because $Dt^*(K_{W'}) = 2K_W t^*(K_{W'}) + B_0 t^*(K_{W'}) = 2(t^*(K_{W'}) + E)t^*(K_{W'}) + B_0 t^*(K_{W'}) = 2K_{W'}^2 + B_0 t^*(K_{W'}) \geq 2$ since $K_{W'}^2 > 0$ and $K_{W'}$ is nef.

So, by the algebraic index theorem, $0 \geq (7t^*(K_W) - D)^2 = 49t^*(K_{W'})^2 - 14Dt^*(K_{W'}) + D^2 = 49K_{W'}^2 - 28 + 14$. Thus $K_{W'}^2 \leq 0$, which give a contradiction.
(iv) Since $K_V^2 = K_S^2 - k = 0$, $K_W^2 \geq 0$. $K_W D = 4$ yields $K_W^2 \leq 1$. $K_W^2 \geq 0$ and $K_W D = 4$ means that W is not birational to an Enriques surface. Again $k = 7$ implies that if W is a rational surface then $K_W^2 = 0$. But then $h^0(W, \mathcal{O}_W(-K_W)) > 0$ and this is impossible because D is nef.

If W is properly elliptic then $K_W^2 = 0$. And if $K_W^2 = 1$ then W is a minimal surface of general type by Lemma 3.4.

Now suppose that $K_W^2 = 0$ and W is of general type. Then we consider a birational morphism $t\colon\ W \to W'$ such that W' is the minimal model of W. Suppose $K_{W'}^2 \geq 2$.

We write $K_W \equiv t^*(K_{W'}) + E$, $E > 0$. Firstly, $Dt^*(K_{W'}) \leq 4$ because $K_W D = 4$. Secondly, $Dt^*(K_{W'}) \geq 4$: $Dt^*(K_{W'}) = 2K_W t^*(K_{W'}) +$

$B_0t^*(K_{W'}) = 2(t^*(K_{W'})+E)t^*(K_{W'})+B_0t^*(K_{W'}) = 2K_{W'}^2+B_0t^*(K_{W'}) \geq 4$ since we suppose $K_{W'}^2 \geq 2$ and $K_{W'}$ is nef.

Therefore $Dt^*(K_{W'}) = 4$. Then by the algebraic index theorem and $D^2 = 14$, $0 \geq (7t^*(K_W) - 2D)^2 = 49t^*(K_{W'})^2 - 28Dt^*(K_{W'}) + 4D^2 = 49K_{W'}^2 - 112 + 56$, which give a contradiction.

(v) Since $K_V^2 = 2$, $K_W^2 \geq 2$ and so W is either a rational surface or a surface of general type. But if it is a rational surface then $h^0(W, \mathcal{O}_W(-K_W)) > 0$ gives a contradiction. Also $K_WD = 6$ and the algebraic index theorem implies that $K_W^2 \leq 2$.

Now we know that W is of general type with $K_W^2 = 2$, it is enough to prove that W is minimal. Suppose W is not minimal. Then we consider a birational morphism $t\colon W \to W'$ such that W' is the minimal model of W. Also, we can write $K_W \equiv t^*(K_{W'}) + E$, $E > 0$. Firstly, $Dt^*(K_{W'}) \leq 6$ because $K_WD = 6$, and $K_{W'}^2 \geq 3$. Secondly, $Dt^*(K_{W'}) \geq 6$: $Dt^*(K_{W'}) = 2K_Wt^*(K_{W'})+B_0t^*(K_{W'}) = 2(t^*(K_{W'})+E)t^*(K_{W'})+B_0t^*(K_{W'}) = 2K_{W'}^2+B_0t^*(K_{W'}) \geq 6$ since $K_{W'}^2 \geq 3$ and $K_{W'}$ is nef.

Therefore $Dt^*(K_{W'}) = 6$. Then by the algebraic index theorem and $D^2 = 14$, $0 \geq (7t^*(K_{W'}) - 3D)^2 = 49t^*(K_{W'})^2 - 42Dt^*(K_{W'}) + 9D^2 = 49K_{W'}^2 - 252 + 126$, which give a contradiction. □

3.1. **Possibilities for B_0 and W.** We study the possibilities of an irreducible component $\Gamma \subset B_0$ for each number of isolated fixed points. Let Γ_V be the preimage of Γ in the double cover V of W. We do not consider the case $k = 11$ because it is already studied in [1] and [11].

Lemma 3.6. *For any irreducible component $\Gamma \subset B_0$ on W, $2K_V\Gamma_V = \Gamma D$, where $\tilde{\pi}^*\Gamma \equiv 2\Gamma_V$.*

Proof. We have $2\Gamma D = \tilde{\pi}^*(\Gamma)\tilde{\pi}^*(D) = 2\Gamma_V\epsilon^*(2K_S)$. So $\Gamma D = \Gamma_V\epsilon^*(2K_S)$. On the other hand, we know that $\Gamma_V\epsilon^*(2K_S) = 2K_V\Gamma_V$ because $\Gamma_V \cap$ (Exceptional locus of ϵ) $= \varnothing$. So $2K_V\Gamma_V = \Gamma D$. □

Remark 3.7. By Lemma 3.6, ΓD should be even and if $\Gamma D = 0$ then Γ is a (-4)-curve.

3.1.1. **Classification of B_0 for $k = 9$.** In this case, $B_0D = 10$ because $B_0D = (D - 2K_W)D = 14 - 4 = 10$. So $\Gamma D = 10, 8, 6, 4$, or 2.

1) The case $\Gamma D = 10$. Since $D^2 = 14$ and D is nef and big, $0 \geq (7\Gamma - 5D)^2 = 49\Gamma^2 - 70\Gamma D + 25D^2 = 49\Gamma^2 - 350$ by the algebraic index theorem. That is, $\Gamma^2 \leq 7$. Thus we get $\Gamma_V^2 \leq 3$ because $2\Gamma_V^2 = \Gamma^2$. Moreover, we know that $0 \leq p_a(\Gamma_V) = 1 + \frac{1}{2}(\Gamma_V^2 + K_V\Gamma_V) = 1 + \frac{1}{2}(\Gamma_V^2 + 5)$ by Lemma 3.6. Thus $-7 \leq \Gamma_V^2 \leq 3$. By the genus formula, $\Gamma_V^2 = -7, -5, -3, -1, 1, 3$.

(1) The case $\Gamma_V^2 = -7$: In this case, $p_a(\Gamma_V) = 0$. So $\Gamma\colon (0, -14)$. Then if we write that $B_0 = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_l$ such that $\Gamma_0 = \Gamma$ and Γ_i are (-4)-curves for each $i = 1, \ldots, l$, then

$$6 = 2 - 2K_W^2 = K_W(D - 2K_W) = K_WB_0 = 12 + 2l.$$

We get a contradiction because $l = -3$

(2) The cases $\Gamma_V^2 = -5, -3$: By a similar argument as the case (1), we get contradictions because $l < 0$.

(3) The case $\Gamma_V^2 = -1$: We get $p_a(\Gamma_V) = 3$. So $\Gamma\colon (3, -2)$ and $l = 0$.

(4) The case $\Gamma_V^2 = 1$: Here, $p_a(\Gamma_V) = 4$. So Γ: $(4, 2)$ and $l = 1$.
(5) The case $\Gamma_V^2 = 3$: Lastly, $p_a(\Gamma_V) = 5$. So Γ: $(5, 6)$ and $l = 2$.

Now, we have the following possibilities of B_0 in the case $\Gamma D = 10$.

$$B_0 : \underset{(5,6)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}, \underset{(4,2)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1}, \underset{(3,-2)}{\Gamma_0}$$

Remark 3.8. $\underset{(5,6)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$ cannot occur by Proposition 2.1.1 of [13] because a smooth rational curve in B_0 corresponds to a smooth rational curve on S.

2) The case $\Gamma_0 D = 8$ and $\Gamma_1 D = 2$. We can use the similar argument as the above 3.1.1.1) for each of $\Gamma_0 D$ and $\Gamma_1 D$. However, we have to consider $B_0 = \Gamma_0 + \Gamma_1 + \Gamma_1' + \cdots + \Gamma_l'$ to get the possibilities for B_0, where Γ_i': $(0, -4)$ for all $i \in \{1, 2, \ldots, l\}$ if it exists. Then we get the following possible cases.

$$B_0 : \underset{(4,4)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}, \underset{(4,4)}{\Gamma_0} + \underset{(0,-6)}{\Gamma_1}, \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$$

Now, we give all remaining cases by the similar argument as the above 3.1.1.2).

3) The case $\Gamma_0 D = 6$ and $\Gamma_1 D = 4$.

$$B_0 : \underset{(3,2)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}, \underset{(3,2)}{\Gamma_0} + \underset{(1,-4)}{\Gamma_1}, \underset{(2,-2)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1}$$

4) The case $\Gamma_0 D = 6$, $\Gamma_1 D = 2$ and $\Gamma_2 D = 2$.

$$B_0 : \underset{(3,2)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$$

5) The case $\Gamma_0 D = 4$, $\Gamma_1 D = 4$ and $\Gamma_2 D = 2$.

$$B_0 : \underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$$

6) The case $\Gamma_0 D = 4$, $\Gamma_1 D = 2$, $\Gamma_2 D = 2$ and $\Gamma_3 D = 2$.
We get a contradiction by the similar argument in 3.1.1.1).(1).
7) The case $\Gamma_0 D = 2$, $\Gamma_1 D = 2$, $\Gamma_2 D = 2$, $\Gamma_3 D = 2$ and $\Gamma_4 D = 2$.
This case is also ruled out by the similar argument in 3.1.1.1).(1).

By Theorem 3.5 and from the above classification, we get the following table:

3.1.2. **Classification of B_0 for $k = 7$.** In this case, $B_0 D = 6$. So ΓD can be $6, 4, 2$. By using similar arguments as the above 3.1.1, we get the following tables related to K_W^2 and B_0 for each case of ΓD.
1) The case $\Gamma D = 6$.

K_W^2	B_0
1	$\underset{(3,2)}{\Gamma_0}$
0	$\underset{(3,2)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1}, \underset{(2,-2)}{\Gamma_0}$

Lemma 3.9. $B_0 = \underset{(3,2)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1}$ *is not possible.*

Proof. Now, we know that W is minimal properly elliptic, or of general type whose the minimal model has $K^2 = 1$ by Theorem 3.5. If W is minimal properly elliptic, then we get a contradiction by Miyaoka's theorem in [13] because W has seven disjoint (-2)-curves and one (-4)-curve.

Table 1: Classifications of K_W^2, B_0 and W for $k = 9$

K_W^2	B_0	W
-2	$\underset{(4,2)}{\Gamma_0} + \underset{(0,-4)}{\Gamma_1}$	$\kappa(W) \leq 1$
	$\underset{(3,-2)}{\Gamma_0}$	
	$\underset{(4,4)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$	
	$\underset{(4,4)}{\Gamma_0} + \underset{(0,-6)}{\Gamma_1}$	
	$\underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$	
	$\underset{(3,2)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(0,-4)}{\Gamma_2}$	
	$\underset{(3,2)}{\Gamma_0} + \underset{(1,-4)}{\Gamma_1}$	
	$\underset{(2,-2)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1}$	
	$\underset{(3,2)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$	
	$\underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$	

So, suppose that W is of general type whose minimal model has $K^2 = 1$. Then we consider a birational morphism $t : W \to W'$ such that W' is the minimal model of W. We write $K_W \equiv t^*(K_{W'}) + E$, where E is the unique (-1)-curve. Firstly, E cannot meet seven disjoint N_i because $K_{W'} t(N_i) = -N_i E$ and $K_{W'}$ is nef. And $\Gamma_1 E \leq 1$ because $K_W B_0 = 4$, $K_W \Gamma_0 = 2$, and $t^*(K_{W'})\Gamma_1 \geq 1$. Then, Miyaoka's theorem [13] again gives a contradiction because W' has seven disjoint (-2)-curves, and one (-4)-curve or one (-3)-curve. $\square$

2) The case $\Gamma_0 D = 4$ and $\Gamma_1 D = 2$.

K_W^2	B_0
0	$\underset{(2,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$

3) The case $\Gamma_0 D = 2$, $\Gamma_1 D = 2$ and $\Gamma_2 D = 2$.

This case is not possible by the similar argument in 3.1.1.1).(1).

Table 2: Classifications of K_W^2, B_0 and W for $k = 7$

K_W^2	B_0	W
1	$\underset{(3,2)}{\Gamma_0}$	minimal of general type
0	$\underset{(2,-2)}{\Gamma_0}$	minimal properly elliptic, or of general
	$\underset{(2,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$	type whose the minimal model has $K^2 = 1$

3.1.3. **Classification of B_0 for $k = 5$.** In this case, $B_0 D = 2$. So ΓD can be 2. By using similar arguments as the above 3.1.1, we get the following table relating to K_W^2 and B_0 for ΓD.

Table 3: Classifications of K_W^2, B_0 and W for $k = 5$

K_W^2	B_0	W
2	$\underset{(1,-2)}{\Gamma_0}$	of general type

4. Quotient surface birational to an Enriques surface

In this section we study the case when W is birational to an Enriques surface.

Theorem 4.1. *Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 7$ having an involution σ. If W is birational to an Enriques surface then $k = 9, K_W^2 = -2$, and the branch divisor $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ or $\underset{(3,-2)}{\Gamma_0}$. Furthermore, S has a 2-torsion element.*

Suppose W is birational an Enriques surface. Then by Theorem 3.5, we have $k = 9$ and $K_W^2 = -2$. Consider the contraction maps:

$$W \xrightarrow{\varphi_1} W_1 \xrightarrow{\varphi_2} W',$$

where $\bar{E}_1$ is (-1)-curve on W, $\bar{E}_2$ is (-1)-curve on W_1, φ_i is the contraction of the (-1)-curve $\bar{E}_i$, and W' is an Enriques surface.

Lemma 4.2. *i) $N_i \cap \bar{E}_1 \neq \varnothing$ for some $i \in \{1, 2, \dots, 9\}$.*
ii) $N_1\bar{E}_1 = 1$ after relabeling $\{N_1, \dots, N_9\}$.
iii) $N_s\bar{E}_1 = 0$ for all $s \in \{2, \dots, 9\}$.

Proof. i) Suppose that $N_i \cap \bar{E}_1 = \varnothing$ for all $i = 1, \dots, 9$. Let A be the number of disjoint (-2)-curves on W_1. Then by Lemma 3.4 (i), $9 \leq A \leq \rho(W_1) - 2 = 9$. Thus $A = 9$ and W_1 should be a minimal surface by Lemma 3.4 (ii). This is a contradiction because W_1 is not minimal. Hence $N_i \cap \bar{E}_1 \neq \varnothing$ for some $i \in \{1, 2, \dots, 9\}$.
ii) By part i) we may choose a (-2)-curve N_1 such that $N_1\bar{E}_1 = \alpha > 0$. Then $(\varphi_1(N_1))^2 = -2 + \alpha^2$ and $\varphi_1(N_1)K_{W_1} = -\alpha$. We claim that α must be 1. Indeed, suppose $\alpha \geq 2$, then $(\varphi_1(N_1))^2 > 0$, so $\varphi_2 \circ \varphi_1(N_1)$ is a curve on W'. Moreover, $\varphi_2 \circ \varphi_1(N_1)K_{W'} \leq \varphi_1(N_1)K_{W_1}$. But the left side is zero because $2K_{W'} \equiv 0$ and the right side is negative because $\varphi_1(N_1)K_{W_1} = -\alpha$ by our assumption. This is a contradiction, thus $\alpha = 1$.
iii) Suppose that $N_s\bar{E}_1 \neq 0$ for some $s \in \{2, \dots, 9\}$. Then W_1 would contain a pair of irreducible (-1)-curves with nonempty intersection. This is impossible because $K_{W'}$ is nef. Hence $N_s\bar{E}_1 = 0$ for all $s \in \{2, \dots, 9\}$. □

In this situation, consider an irreducible nonsingular curve Γ disjoint to N_1 and such that $\bar{E}_1\Gamma = \beta$. Then we obtain the following.

Lemma 4.3. $2p_a(\Gamma) - 2 = \Gamma^2 + 2\beta$.

Proof. By Lemma 4.2,

$$K_W \equiv \varphi_1^*(K_{W_1}) + \bar{E}_1 \equiv \varphi_1^*(\varphi_2^*(K_{W'}) + \bar{E}_2) + \bar{E}_1 \equiv \varphi_1^* \circ \varphi_2^*(K_{W'}) + N_1 + 2\bar{E}_1.$$

So $K_W\Gamma = \varphi_1^* \circ \varphi_2^*(K_{W'})\Gamma + N_1\Gamma + 2\bar{E}_1\Gamma = 2\beta$ since $2K_{W'} \equiv 0$ and N_1 and Γ are disjoint. Thus we get $2p_a(\Gamma) - 2 = \Gamma^2 + 2\beta$. □

By referring to Table 1. of Section 3.1.1 with respect to $K_W^2 = -2$ and $k = 9$, we obtain a list of possible branch curves B_0. Then we can consider Γ as one of the components Γ_i in the B_0. The possibilities for Γ which we will consider are:

$$(0,-4),(2,-2),(2,0),(1,-2),(0,-6),(3,2),(1,-4).$$

We treat each case separately.

a) The case $\Gamma\colon (0,-4)$
By Lemma 4.3 (i), $\beta = 1$. Thus W' should contain disjoint 9 curves of type $(0,-2)$. This is a contradiction because W' can contain at most eight disjoint (-2)-curves which are $(0,-2)$ since it is an Enriques surface.

From now on, we consider the nodal Enriques surface Σ' obtained by contracting eight (-2)-curves $\tilde{N}_i$, $i = 2,\dots,9$, where $\tilde{N}_i := \varphi_2 \circ \varphi_1(N_i)$ on W'. The surface Σ' has eight nodes q_i, $i = 2,\dots,9$ and $\tilde{\Gamma}_{\Sigma'}$ which is image of $\tilde{\Gamma}$, where $\tilde{\Gamma} := \varphi_2 \circ \varphi_1(\Gamma)$ on W'. By Theorem 4.1 in [12], $\Sigma' = D_1 \times D_2/G$, where D_1 and D_2 are elliptic curves and G is a finite group $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$. Let p be the quotient map $D_1 \times D_2 \longrightarrow D_1 \times D_2/G = \Sigma'$. The map p is étale outside the preimage of nodes q_i on Σ', and we note that $\tilde{\Gamma}_{\Sigma'}$ does not meet with any eight nodes q_i on Σ'. We write $\hat{\Gamma}_{D_1\times D_2}$ for a component of $p^{-1}(\tilde{\Gamma}_{\Sigma'})$.

b) The case $\Gamma\colon (0,-6)$
By Lemma 4.3, $\beta = 2$. So $\tilde{\Gamma}$ is (2,2). Then the normalization $\hat{\Gamma}^{nor}$ of $\hat{\Gamma}_{D_1\times D_2}$ is a smooth rational curve since $p_a(\Gamma) = 0$ and Γ is smooth.
Let pr_i be the projection map $D_1 \times D_2 \longrightarrow D_i$. Then this induces morphisms $p_i\colon \hat{\Gamma}^{nor} \longrightarrow D_i$ which factors through pr_i. Then since $\hat{\Gamma}_{D_1\times D_2}$ is a curve on $D_1 \times D_2$, p_i should be a surjective morphism for some $i \in \{1,2\}$. However, this is impossible because $p_g(\hat{\Gamma}^{nor}) = 0$ and $p_g(D_i) = 1$.

c) The case $\Gamma\colon (1,-4)$
By Lemma 4.3, $\beta = 2$. So $\tilde{\Gamma}$ is (3,4). Then the normalization $\hat{\Gamma}^{nor}$ of $\hat{\Gamma}_{D_1\times D_2}$ is a smooth elliptic curve because $p_a(\Gamma) = 1$ and Γ is smooth. Thus $\hat{\Gamma}^{nor} \longrightarrow D_1 \times D_2$ is a morphism of Abelian varieties and so must be linear, which implies that $\hat{\Gamma}_{D_1\times D_2}$ is smooth. Thus $\tilde{\Gamma}_{\Sigma'}$ is also smooth because $\tilde{\Gamma}_{\Sigma'}$ does not meet with any eight nodes q_i on Σ' and p is étale on away from the nodes q_i. This is a contradiction since we assumed $\tilde{\Gamma}_{\Sigma'}$ to be singular.

d) The case $\underset{(3,2)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$
By Lemma 4.3, we have $\bar{E}_1\Gamma_i = 1$ for $i = 0,1,2$. So we get $\tilde{\Gamma}_0\colon (3,4)$, $\tilde{\Gamma}_1\colon (1,0)$, $\tilde{\Gamma}_2\colon (1,0)$ and $\tilde{\Gamma}_i\tilde{\Gamma}_j = 2$ for $i \neq j$ on the Enriques surface W'. Now, we apply Proposition 3.1.2 of [4] to the curve $\tilde{\Gamma}_2$. Then one of the linear systems $|\tilde{\Gamma}_2|$ or $|2\tilde{\Gamma}_2|$ gives an elliptic fibration $f\colon W' \longrightarrow \mathbb{P}^1$. So we have the reducible non-multiple degenerate fibres $\tilde{T}_1(= \tilde{N}_2+\tilde{N}_3+\tilde{N}_4+\tilde{N}_5+2E_1)$ and $\tilde{T}_2(= \tilde{N}_6+\tilde{N}_7+\tilde{N}_8+\tilde{N}_9+2E_2)$ of f by Theorem 5.6.2 of [4], since W' has eight disjoint (-2)-curves. Moreover, f has two double fibres $2F_1$ and $2F_2$ since W' is an Enriques surface.

(1) Suppose $|\tilde{\Gamma}_2|$ determines the elliptic fibration. Then $\tilde{\Gamma}_2$ is a fibre of f. Since $\tilde{\Gamma}_1\tilde{\Gamma}_2 = 2$ (they meet at a point with multiplicity 2), $2F_1\tilde{\Gamma}_1 = 2$, $2F_2\tilde{\Gamma}_1 = 2$ and $\tilde{T}_i\tilde{\Gamma}_1 = 2$ for $i = 1, 2$, we apply Hurwitz's formula to the covering $f|_{\tilde{\Gamma}_1} : \tilde{\Gamma}_1 \longrightarrow \mathbb{P}^1$ to obtain

$$0 = 2p_g(\tilde{\Gamma}_1) - 2 \geq 2(-2) + 5 = 1$$

which is impossible.

(2) Suppose $|2\tilde{\Gamma}_2|$ determines the elliptic fibration. Then $2\tilde{\Gamma}_2$ is a fibre of f. Since $2F_1\tilde{\Gamma}_1 (= 2\tilde{\Gamma}_2\tilde{\Gamma}_1) = 4$, $2F_2\tilde{\Gamma}_1 = 4$ and $\tilde{T}_i\tilde{\Gamma}_1 = 4$ for $i = 1, 2$, we apply Hurwitz's formula to the covering $f|_{\tilde{\Gamma}_1} : \tilde{\Gamma}_1 \longrightarrow \mathbb{P}^1$ to obtain

$$0 = 2p_g(\tilde{\Gamma}_1) - 2 \geq 4(-2) + 3 + 2 + 2 + 2 = 1,$$

which is impossible.

e) The case $\underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$

By Lemma 4.3, $\bar{E}_1\Gamma_i = 1$ for $i = 0, 1, 2$. So we have $\tilde{\Gamma}_0 : (2, 2)$, $\tilde{\Gamma}_1 : (2, 2)$, $\tilde{\Gamma}_2 : (1, 0)$ and $\tilde{\Gamma}_i\tilde{\Gamma}_j = 2$ for $i \neq j$ on the Enriques surface W'.

Lemma 4.4. $h^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1)) = 2$.

Proof. Since $2K_{W'} \equiv 0$ and $K_{W'} + \tilde{\Gamma}_1$ is nef and big,

$$\begin{aligned} h^i(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1)) &= h^i(W', \mathcal{O}_{W'}(2K_{W'} + \tilde{\Gamma}_1)) \\ &= h^i(W', \mathcal{O}_{W'}(K_{W'} + (K_{W'} + \tilde{\Gamma}_1))) \\ &= 0 \end{aligned}$$

for $i = 1, 2$ by Kawamata-Viehweg Vanishing Theorem. Thus

$$h^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1)) = 2$$

by Riemann-Roch Theorem. □

Lemma 4.5. *Let T be a nef and big divisor on W'. Then any divisor U in a linear system $|T|$ is connected.*

Proof. Consider an exact sequence

$$0 \longrightarrow \mathcal{O}_{W'}(-U) \longrightarrow \mathcal{O}_{W'} \longrightarrow \mathcal{O}_U \longrightarrow 0.$$

Then we get $H^0(\mathcal{O}_{W'}) \cong H^0(\mathcal{O}_U)$ by the long exact sequence for cohomology, and so U is connected. □

Now, we apply Proposition 3.1.2 of [4] to the curve $\tilde{\Gamma}_2$. Then one of the linear systems $|\tilde{\Gamma}_2|$ or $|2\tilde{\Gamma}_2|$ gives an elliptic fibration $f : W' \longrightarrow \mathbb{P}^1$. So we have the reducible non-multiple degenerate fibres $\tilde{T}_1 (= \tilde{N}_2 + \tilde{N}_3 + \tilde{N}_4 + \tilde{N}_5 + 2E_1)$, $\tilde{T}_2 (= \tilde{N}_6 + \tilde{N}_7 + \tilde{N}_8 + \tilde{N}_9 + 2E_2)$ and two double fibres $2F_1$, $2F_2$ of the fibration f.

(1) Suppose $|\tilde{\Gamma}_2|$ determines the elliptic fibration. Consider an exact sequence $0 \longrightarrow \mathcal{O}_{W'}(\tilde{\Gamma}_1 - E_1) \longrightarrow \mathcal{O}_{W'}(\tilde{\Gamma}_1) \longrightarrow \mathcal{O}_{E_1}(\tilde{\Gamma}_1) \longrightarrow 0$. If we assume $H^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1 - E_1)) \neq 0$, then $\tilde{\Gamma}_1 \equiv 2E_1 + \tilde{N}_2 + \tilde{N}_3 + \tilde{N}_4 + \tilde{N}_5 + G \equiv \tilde{\Gamma}_2 + G$ for some effective divisor G, and so $p_a(G) = 0$ because $\tilde{\Gamma}_2 G = 2$. So there is an irreducible smooth

curve C with $p_a(C)=0$ (i.e. C is an irreducible (-2)-curve) as a component of G. We claim $C\tilde{N}_i=0$ for $i=2,3,\dots,9$. Indeed, suppose $C\tilde{N}_i>0$ for some i, and then $0=G\tilde{N}_i=(H+C)\tilde{N}_i$, where $G=H+C$ for some effective divisor H. Since $H\tilde{N}_i<0$, $\tilde{N}_i$ is a component of H. Thus $\tilde{\Gamma}_1-\tilde{\Gamma}_2\equiv G=\tilde{N}_i+I$ for some effective divisor I, which is impossible by $p_a(\tilde{\Gamma}_1)=2$, $p_a(\tilde{\Gamma}_2)=1$, $\tilde{\Gamma}_2 I=2$, $\tilde{N}_i I=2$ and connectedness among $\tilde{\Gamma}_2,\tilde{N}_i$ and I induced from Lemma 4.5 since $\tilde{\Gamma}_1$ is nef and big. On the other hand, suppose $C\tilde{N}_i<0$ for some i, then $C=\tilde{N}_i$ because C and $\tilde{N}_i$ are irreducible and reduced. Thus $\tilde{\Gamma}_1-\tilde{\Gamma}_2\equiv G=\tilde{N}_i+H$ for an effective divisor H, which is impossible by $p_a(\tilde{\Gamma}_1)=2$, $p_a(\tilde{\Gamma}_2)=1$, $\tilde{\Gamma}_2 H=2$ and $\tilde{N}_i H=2$ and connectedness among $\tilde{\Gamma}_2,\tilde{N}_i$ and H induced from Lemma 4.5 since $\tilde{\Gamma}_1$ is nef and big. Hence we have nine disjoint (-2)-curves $C,\tilde{N}_2,\dots,\tilde{N}_9$, which induce a contradiction on the Enriques surface W' by Lemma 3.4. Now, we have $H^0(W',\mathcal{O}_{W'}(\tilde{\Gamma}_1-E_1))=0$, and so

$$H^0(W',\mathcal{O}_{W'}(\tilde{\Gamma}_1))\longrightarrow H^0(E_1,\mathcal{O}_{E_1}(\tilde{\Gamma}_1))$$

is an injective map.
Since $h^0(W',\mathcal{O}_{W'}(\tilde{\Gamma}_1))=2$ and $h^0(E_1,\mathcal{O}_{E_1}(\tilde{\Gamma}_1))=2$ (because $\tilde{\Gamma}_1E_1=1$), $\tilde{\Gamma}_1\equiv\tilde{N}_2+\tilde{N}_3+\tilde{N}_4+\tilde{N}_5+L$ for some effective divisor L, which is impossible by $p_a(\tilde{\Gamma}_1)=2$ and $\tilde{N}_iL=2$ for all $i=2,3,4,5$ and connectedness among $\tilde{N}_2,\tilde{N}_3,\tilde{N}_4,\tilde{N}_5$ and L induced from Lemma 4.5 since $\tilde{\Gamma}_1$ is nef and big.

(2) Suppose $|2\tilde{\Gamma}_2|$ determines the elliptic fibration. Consider an exact sequence

$$0\longrightarrow\mathcal{O}_{W'}(\tilde{\Gamma}_1-E_1)\longrightarrow\mathcal{O}_{W'}(\tilde{\Gamma}_1)\longrightarrow\mathcal{O}_{E_1}(\tilde{\Gamma}_1)\longrightarrow 0.$$

If we assume $H^0(W',\mathcal{O}_{W'}(\tilde{\Gamma}_1-E_1))\neq 0$, then $\tilde{\Gamma}_1\equiv E_1+\tilde{N}_2+\tilde{N}_3+\tilde{N}_4+\tilde{N}_5+G$ for some effective divisor G, which is impossible by $p_a(\tilde{\Gamma}_1)=2$, $E_1G=0$ and $\tilde{N}_iG=1$ for all $i=2,3,4,5$ and connectedness among $E_1,\tilde{N}_2,\tilde{N}_3,\tilde{N}_4,\tilde{N}_5$ and G induced from Lemma 4.5 since $\tilde{\Gamma}_1$ is nef and big. Thus we have $H^0(W',\mathcal{O}_{W'}(\tilde{\Gamma}_1-E_1))=0$, and so

$$H^0(W',\mathcal{O}_{W'}(\tilde{\Gamma}_1))\longrightarrow H^0(E_1,\mathcal{O}_{E_1}(\tilde{\Gamma}_1))$$

is an injective map.
Since $h^0(W',\mathcal{O}_{W'}(\tilde{\Gamma}_1))=2$ and $h^0(E_1,\mathcal{O}_{E_1}(\tilde{\Gamma}_1))=3$ (because $\tilde{\Gamma}_1E_1=2$), $\tilde{\Gamma}_1\equiv\tilde{N}_2+\tilde{N}_3+\tilde{N}_4+\tilde{N}_5+L$ for some effective divisor L, which is also impossible by $p_a(\tilde{\Gamma}_1)=2$ and $\tilde{N}_iL=2$ for all $i=2,3,4,5$ and connectedness among $\tilde{N}_2,\tilde{N}_3,\tilde{N}_4,\tilde{N}_5$ and L induced from Lemma 4.5 since $\tilde{\Gamma}_1$ is nef and big.

f) The case $\underset{(2,-2)}{\Gamma_0}+\underset{(2,0)}{\Gamma_1}$
By Lemma 4.3 (i), $\bar{E}_1\Gamma_0=2$ and $\bar{E}_1\Gamma_1=1$. So we have $\tilde{\Gamma}_0\colon(4,6)$ and $\tilde{\Gamma}_1\colon(2,2)$ on the Enriques surface W'.

Consider an elliptic fibration of Enriques surface $f\colon W'\longrightarrow\mathbb{P}^1$, and assume $\tilde{\Gamma}_1F=2\gamma$, where F is a general fibre of f. Then $\gamma>0$

because $\tilde{\Gamma}_1$ cannot occur in a fibre of f by $p_a(\tilde{\Gamma}_1) = 2$. Moreover, consider an exact sequence

$$0 \longrightarrow \mathcal{O}_{W'}(\tilde{\Gamma}_1 - E_1) \longrightarrow \mathcal{O}_{W'}(\tilde{\Gamma}_1) \longrightarrow \mathcal{O}_{E_1}(\tilde{\Gamma}_1) \longrightarrow 0.$$

If we assume $H^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1 - E_1)) \neq 0$, then $\tilde{\Gamma}_1 \equiv E_1 + \tilde{N}_2 + \tilde{N}_3 + \tilde{N}_4 + \tilde{N}_5 + G$ for some effective divisor G, which is impossible by $p_a(\tilde{\Gamma}_1) = 2$, $\tilde{N}_i G = 1$ for all $i = 2, 3, 4, 5$ and connectedness among $E_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4, \tilde{N}_5$ and G induced from Lemma 4.5 since $\tilde{\Gamma}_1$ is nef and big. Now, we have $H^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1 - E_1)) = 0$, and so

$$H^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1)) \longrightarrow H^0(E_1, \mathcal{O}_{E_1}(\tilde{\Gamma}_1))$$

is an injective map. Since $h^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_1)) = 2$ by Lemma 4.4 and $h^0(E_1, \mathcal{O}_{E_1}(\tilde{\Gamma}_1)) = \gamma + 1$ (because $\tilde{\Gamma}_1 E_1 = \gamma$), $\tilde{\Gamma}_1 \equiv \tilde{N}_2 + \tilde{N}_3 + \tilde{N}_4 + \tilde{N}_5 + L$ for some effective divisor L, which is impossible by $p_a(\tilde{\Gamma}_1) = 2$ and $\tilde{N}_i L = 2$ for all $i = 2, 3, 4, 5$ and connectedness among $\tilde{N}_2, \tilde{N}_3, \tilde{N}_4, \tilde{N}_5$ and L induced from Lemma 4.5 since $\tilde{\Gamma}_1$ is nef and big.

Therefore, all other cases except $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ or $\underset{(3,-2)}{\Gamma_0}$ are excluded.

Lemma 4.6. *If W is birational to an Enriques surface then S has a 2-torsion element.*

Proof. If W is birational to an Enriques surface then $2K_W$ can be written as $2A$ where A is an effective divisor. Thus $2K_V \equiv \tilde{\pi}^*(2A) + 2\tilde{R}$, where $\tilde{R}$ is the ramification divisor of $\tilde{\pi}$. So $G = \tilde{\pi}^*(A) + \tilde{R}$ is an effective divisor such that $G \sim K_V$ but $G \not\equiv K_V$ because G is effective and $p_g(V) = 0$. Since $2G \equiv 2K_V$, $G - K_V$ is a 2-torsion element, and so S has a 2-torsion element. □

Remark 4.7. Suppose $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$. By Lemma 4.3, $\bar{E}_1 \Gamma_0 = 2$ and $\bar{E}_1 \Gamma_1 = 1$. So we have $\tilde{\Gamma}_0 \colon (5, 8)$, $\tilde{\Gamma}_1 \colon (1, 0)$ and $\tilde{\Gamma}_0 \tilde{\Gamma}_1 = 4$ on the Enriques surface W'. We have $h^0(W', \mathcal{O}_{W'}(\tilde{\Gamma}_0)) = 5$ since $\tilde{\Gamma}_0 \colon (5, 8)$. However, the intersection number $\tilde{\Gamma}_0 \tilde{\Gamma}_1 = 4$ together with tangency condition gives a six dimensional conditions.

By the results in Section 3 and 4, we have the table of classification in Introduction.

5. Examples

There is an example of a minimal surface S of general type with $p_g(S) = q(S) = 0$, $K_S^2 = 7$ with an involution. Such an example can be found in Example 4.1 of [10]. Since the surface S is constructed by bidouble cover (i.e. $\mathbb{Z}_2^2$-cover), there are three involutions γ_1, γ_2 and γ_3 on S. The bicanonical map φ is composed with the involution γ_1 but not with γ_2 and γ_3. Thus the pair (S, γ_1) has $k = 11$ by Proposition 3.2, and then W_1 is rational and $K_{W_1}^2 = -4$ by Theorem 3.5 (*ii*), where W_1 is the blow-up of all the nodes in S/γ_1. On the other hand, we cannot see directly about k, K^2 and the Kodaira dimension of the quotients in the case (S, γ_2) and (S, γ_3). We use notations of Example 4.1 of [10], but P denotes Σ of Example 4.1 of [10].

Moreover, W_i comes from the blow-ups at all the nodes of $\Sigma_i := S/\gamma_i$ for $i = 1, 2, 3$.

Now, we observe that W_i is constructed by using a double covering T_i of a rational surface P with a branch divisor related to L_i. The surface P is obtained as the blow-up of six points on a configuration of lines in $\mathbb{P}^2$. The surface W_i is obtained by examining (-1) and (-2)-curves on T_i and contracting some of them.

We will now explain this examination in more details for each case. Firstly, for $i = 1$, then $K_{T_1}^2 = -6$ since $K_{T_1} \equiv \pi_1^*(K_P + L_1)$, where $\pi_1 \colon T_1 \longrightarrow P$ is the double cover. We observe that there are only two (-1)-curves on T_1 because S_3, S_4 are on the branch locus of π_1. So $K_{W_1}^2 = K_{\Sigma_1}^2 = -6+2 = -4$. On the other hand, we also observe that there are only seven nodes and four (-2)-curves on T_1 because $D_2 D_3 = 7$ and S_1 and S_2 do not contain in $D_2 + D_3$. So Σ_1 has $k = 11$ nodes. Moreover, $H^0(T_1, \mathcal{O}_{T_1}(2K_{T_1})) = H^0(P, \mathcal{O}_P(2K_P + 2L_1)) \oplus H^0(P, \mathcal{O}_P(2K_P + L_1))$ since $2K_{T_1} \equiv \pi_1^*(2K_P + 2L_1)$ and $\pi_{1*}(\mathcal{O}_{T_1}) = \mathcal{O}_P \oplus \mathcal{O}_P(-L_1)$. So $H^0(T_1, \mathcal{O}_{T_1}(2K_{T_1})) = 0$ because $2K_P + 2L_1 = 4l - 2e_2 - 4e_4 - 2e_5 - 2e_6$ and $2K_P + L_1 = -l + e_1 + e_3 - e_4$. This means that T_1 is rational, and therefore W_1 is rational. For the branch divisor B_0, we observe f_2 and Δ_1 in D_1. Since $f_2 D_2 = 4$ and $f_2 D_3 = 4$, $f_2(D_2 + D_3) = 8$. By Hurwitz's formula, $2p_a(\Gamma_0) - 2 = 2(p_a(f_2) - 2) + 8$, and so $p_a(\Gamma_0) = 3$ because f_2 is rational, and moreover $\Gamma_0^2 = 0$ because $f_2^2 = 0$. This means $\Gamma_0 \colon (3, 0)$. Similarly, since $\Delta_1 D_2 = 1$ and $\Delta_1 D_3 = 5$, $\Delta_1(D_2 + D_3) = 6$. By Hurwitz's formula, $2p_a(\Gamma_1) - 2 = 2(p_a(\Delta_1) - 2) + 6$, and so $p_a(\Gamma_1) = 2$ because Δ_1 is rational, and moreover $\Gamma_1^2 = -2$ because $\Delta_1^2 = -1$. This means $\Gamma_1 \colon (2, -2)$, thus $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(2,-2)}{\Gamma_1}$.

Secondly, in the case $i = 2$, we calculate $K_{T_2}^2 = -6$. We observe that there are only four (-1)-curves on T_2 because S_1, S_2, S_3, S_4 are on the branch locus. So $K_{W_2}^2 = K_{\Sigma_2}^2 = -6 + 4 = -2$. On the other hand, we also observe that there are only nine nodes on T_2 because $D_1 D_3 = 9$. So Σ_2 has $k = 9$ nodes. Also, $H^0(T_2, \mathcal{O}_{T_2}(2K_{T_2})) = 0$ by a similar argument as the case $i = 1$. So W_2 is rational. For the branch divisor B_0, we observe f_3 and Δ_2 in D_2. Since $f_3 D_1 = 2$ and $f_3 D_3 = 6$, $p_a(\Gamma_0) = 3$ because f_3 is rational, and $\Gamma_0^2 = 0$ because $f_3^2 = 0$. This means $\Gamma_0 \colon (3, 0)$. Moreover, since $\Delta_2 D_1 = 3$ and $\Delta_2 D_3 = 1$, $p_a(\Gamma_1) = 1$ because Δ_2 is rational, and $\Gamma_1^2 = -2$ because $\Delta_2^2 = -1$. This means $\Gamma_1 \colon (1, -2)$, thus $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$.

Lastly, for $i = 3$, we get $K_{T_3}^2 = -4$. There are only two (-1)-curves on T_3 because S_1, S_2 are on the branch locus. So $K_{W_3}^2 = K_{\Sigma_3}^2 = -4 + 2 = -2$. On the other hand, there are only nine nodes on T_3 because $D_1 D_2 = 5$ and S_3 and S_4 do not contain in $D_1 + D_2$. So Σ_2 has $k = 9$ nodes. Also, $H^0(T_3, \mathcal{O}_{T_3}(2K_{T_3})) = 0$ by a similar argument to the case $i = 1$. So W_3 is rational. For the branch divisor B_0, we observe f_1, f_1' and Δ_3 in D_3. Since $f_1 D_1 = 4$ and $f_1 D_2 = 2$, $p_a(\Gamma_0) = 2$ because f_1 is rational, and $\Gamma_0^2 = 0$ because $f_1^2 = 0$. This means $\Gamma_0 \colon (2, 0)$, and Γ_1 related to f_1' is also of type $(2, 0)$. Moreover, since $\Delta_3 D_1 = 1$ and $\Delta_3 D_2 = 3$, $p_a(\Gamma_2) = 1$ because Δ_3 is rational, and $\Gamma_2^2 = -2$ because $\Delta_3^2 = -1$. This means $\Gamma_2 \colon (1, -2)$, thus $B_0 = \underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$.

The following table summaries our result:

	k	$K^2_{W_i}$	B_0	W_i
(S, γ_1)	11	-4	$\underset{(3,0)}{\Gamma_0} + \underset{(2,-2)}{\Gamma_1}$	rational
(S, γ_2)	9	-2	$\underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$	rational
(S, γ_3)	9	-2	$\underset{(2,0)}{\Gamma_0} + \underset{(2,0)}{\Gamma_1} + \underset{(1,-2)}{\Gamma_2}$	rational

Recently, Rito [15] gave a new example of surface of general type with $p_g = q = 0$ and $K^2 = 7$ by using a double cover of a rational surface. In his example, B_0 is also $\underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$.

Acknowledgements. Both authors would like to thank Margarida Mendes Lopes for sharing her ideas which run through this work. The simplified proof of Theorem 3.5 and Lemma 4.6 are due to her. And they would like to thank Yifan Chen, Stephen Coughlan, JongHae Keum, Miles Reid, and Carlos Rito for some useful comments.

The first author was partially supported by the Special Research Grant of Sogang University. And this work was partially supported by the World Class University program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (R33-2008-000-10101-0).

References

[1] A. Calabri, C. Ciliberto, M. Mendes Lopes, *Numerical Godeaux surfaces with an involution*, Trans. Amer. Math. Soc. **359** (2007), no. 4, 1605–1632.

[2] A. Calabri, M. Mendes Lopes, R. Pardini *Involutions on numerical Campedelli surfaces*, Tohoku Math. J. (2) **60** (2008), no. 1, 1–22.

[3] L. Campedelli, *Sopra alcuni piani doppi notevoli con curva di diramazione del decimo ordine*, Atti Accad. Naz. Lincei **15** (1932), 536–542.

[4] F. Cossec, I. Dolgachev, *Enriques surfaces I*, Birkh. Verlag (1989).

[5] I. Dolgachev, M. Mendes Lopes, R. Pardini, *Rational surfaces with many nodes*, Compositio Math. **132** (2002), no. 3, 349–363.

[6] J. Keum, *Some new surfaces of general type with $p_g = 0$*, preprint, (1988).

[7] J. Keum, *Projective surfaces with many nodes*, arXiv:0908.4372, (2009).

[8] J. Keum, Y. Lee, *Fixed locus of an involution acting on a Godeaux surface*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 2, 205–216.

[9] M. Mendes Lopes, R. Pardini, *A new family of surfaces with $p_g = 0$ and $K^2 = 3$*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 4, 507–531.

[10] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geq 7$*, Bull. London Math. Soc. **33** (2001), no. 3, 265–274.

[11] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geq 7$. II*, Bull. London Math. Soc. **35** (2003), no. 3, 337–343.

[12] M. Mendes Lopes, R. Pardini, *Enriques surfaces with eight nodes*, Math. Z. **241** (2002), no. 4, 673–683.

[13] Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann., **268** (1984), 159–171.

[14] D. Naie, *Surfaces d'Enriques et une construction de surfaces de type général avec $p_g = 0$*, Math. Z. **215** (1994), no. 2, 269–280.

[15] C. Rito, *An algorithm to compute plane algebraic curves*, math.AG/arXiv:0906.3480.

[16] G. Xiao, *Degree of the bicanonical map of a surface of general type*, Amer. J. Math., **112** (1990), 713–737.

Department of Mathematics, Sogang University, Sinsu-dong, Mapo-gu, Seoul 121-742, and Korea Institute for Advanced Study, 207-43 Cheongnyangni-dong, Seoul 130-722, Korea
E-mail address: ynlee@sogang.ac.kr

Department of Mathematics, Sogang University, Sinsu-dong, Mapo-gu, Seoul 121-742, Korea
E-mail address: haushin@sogang.ac.kr